The equivalence between the isomorphism between two objects, the isomorphism of the pushforward, and the isomorphism of the pushback

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This short project is dedicated to proving an important theorem. In summary, it states that we can learn a lot about an object X by considering it's relationships to other objects.

In this document we will denote the morphisms between objects X and Y in a category C by $C(X, Y)$. It is often written as $\mathrm{Hom}_C(X, Y)$ or $\mathrm{Hom}(X, Y)$ if the category is clear from context. We will write maps on the right.

Theorem. *Let X and Y be objects from a category C. Then the following are equivalent:*

1. *The morphism $f : X \to Y$ is an isomorphism;*
2. *For any object Z, the pushforward $f_\star : C(Z, X) \to C(Z, Y)$ is an isomorphism (of sets);*
3. *For any object Z, the pullback $f^\star : C(Y, Z) \to C(X, Z)$ is an isomorphism (of sets).*

Proof. We will prove $1 \Leftrightarrow 3$, then $1 \Leftrightarrow 2$, and finally $2 \Rightarrow 3$. For convenience, we have included relevant commutative diagrams in Figure 1.

($1 \Leftrightarrow 3$) First, assume that f is an isomorphism. In other words, the inverse $f^{-1} : Y \to X$ exists. We want to show that the pullback $f^\star$ is an isomorphism of sets. We will do this by constructing a map and showing it is the inverse of $f^\star$. Consider $(f^{-1})^\star$, the pullback of f^{-1}. Let $g_1 \in C(Y, Z)$. Then

$$g_1 f^\star (f^{-1})^\star = f^{-1} f g_1 = g_1.$$

Similarily, let $g_2 \in C(X, Z)$. Then

$$g_2 (f^{-1})^\star f^\star = f f^{-1} g_2 = g_2.$$

Hence $(f^{-1})^\star = (f^\star)^{-1}$, and $f^\star$ is an isomorphism.

Next, assume that $f^\star$ is an isomorphism for all Z. We want to show that f is an isomorphism. Take $Z = X$, so the pullback is a map between $C(Y, X)$ and $C(X, X)$. Since f is surjective, we can find a $h \in C(Y, X)$ such that $hf^\star = fh = \mathrm{id}_X$. This seems like a good candidate for our inverse for f.

Now consider $Z = Y$, so the pullback is a map between $C(Y, Y)$ and $C(X, Y)$. Since

$$\mathrm{id}_Y f^\star = f \mathrm{id}_Y = f$$

and

$$(hf)f^\star = f(hf) = (fh)f = \mathrm{id}_X f = f,$$

we can deduce that $hf = \mathrm{id}_Y$ as $f^\star$ is injective. Therefore h is an inverse for f, and f is an isomorphism.

(1 $\Leftrightarrow$ 2) The reasoning to this is very similar to above, so we omit it.

(2 $\Rightarrow$ 3) Assume $f_\star$ is an isomorphism. From above, we know that f is an isomorphism, so f^{-1} exists. Consider the pullback $(f^{-1})^\star : C(X,Z) \to C(Y,Z)$. Let $g_1 \in C(Y,Z)$. Then

$$g_1 f^\star (f^{-1})^\star = g_1 f f^{-1} = g.$$

Similarily, let $g_2 \in C(X,Z)$. Then

$$g_2 (f^{-1})^\star f^\star = g_2 f^{-1} f = g.$$

As g_1 and g_2 were arbitary, we have shown that $(f^{-1})^\star = (f^\star)^{-1}$, and therefore $f^\star$ is an isomorphism.

□

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ Z \arrow[u, "g_1"] \arrow[ur, dashed, "g_1 f_\star := g_1 f"'] & \end{tikzcd}$$

(a) The pushforward $f_\star$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "g_2 f_* := f g_2"'] & Y \arrow[d, "g_2"] \\ & Z \end{tikzcd}$$

(b) The pullback $f^\star$.

$$\begin{tikzcd} X & Y \arrow[l, "f^{-1}"'] \\ & Z \arrow[u, "g_3"'] \arrow[ul, dashed, "g_3 (f^{-1})_\star := g_3 f^{-1}"] \end{tikzcd}$$

(c) The pushforward $(f^{-1})_\star$.

$$\begin{tikzcd} X \arrow[d, "g_4"'] & Y \arrow[l, "f^{-1}"'] \arrow[dl, dashed, "g_4 (f^{-1})^\star := f^{-1} g_4"] \\ Z & \end{tikzcd}$$

(d) The pullback $(f^{-1})^\star$.

Figure 1: Commutative diagrams of pushforwards and pullbacks.